APPLICATION OF NUMERICAL TECHNIQUES TO COMPARE THE SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT

In this paper, we studied the numerical techniques for the solution of two dimensional Elliptic partial differential equations such as Laplace's and Poisson's equations. These types of differential equations have specific applications in physical and engineering models. The discrete approximation of both equations is based on finite difference method. In this research, five points finite difference approximation is used for Laplace's and Poisson's equations. To solve the resulting finite difference approximation basic iterative methods: Jacobi, Gauss-Seidel and Successive Over Relaxation (SOR) have been used.

KEYWORDS

Numerical techniques, differential equations, finite approximation and basic iterative method

1. INTRODUCTION

Most real mathematical problems do not have analytical solutions. However, they do have the real answers for each of the problems. To obtain these solutions we can use other methods such as graphical representations or numerical analysis. Numerical analysis is the mathematical method that also consider the accuracy of an approximation. The solution in partial differential equations arising in scientific and engineering problems is very important to conclude the situation. In case of analytic solution of partial differential equation also, numerical values at different intervals of variables are required. Therefore numerical solution of a problem is very useful and important. In this study, we will use numerical analytical to solve the elliptic partial differential equations.

2. METHOLOGY

There is no general approach for the numerical solution of non-linear partial differential equations. For the solution of linear equations, however, numerical methods have been developed and these are simple and efficient when compared to the analytical techniques which can become rather involved, particularly when the geometric boundaries have irregular shapes. Numerical techniques have the advantage that they can be applied to problems having irregularly shaped boundaries and are also easy to program for a digital computer.

Several numerical techniques have been developed for the solution of partial differential equations, but only the finite difference methods have become popular and are more gainfully employed than others.

2.1. Finite Difference Grid on Rectangular Domain

The given region (rectangle ABCD) is divided into smaller rectangles of sides $\delta x = h$ and $\delta y = k$. $u(x,y)$ is the temperature as the function of time and space. The original is taken at the centre of the rectangle and the coordinates axes are drawn. The rectangle is divided into 36 small rectangles. Here there are 49 mesh-points or lattices or nodal points or grid points. The value of the function u are $u_{I,j}, u_{i+1,j}, u_{i+2,j}, \ldots, u_{I,j+1}, u_{I,j+2}, \ldots$ at the mesh points.

We take $u_{I,j}$ at origin $O, u_{i+1,j}, u_{i+2,j}, u_{i+3,j}$, etc. on positive i-axis i.e. OX', $u_{i-1,j}, u_{i-2,j}, u_{i-3,j}$, etc. on negative i-axis i.e. OX', and $u_{I,j+1}, u_{I,j+2}, u_{I,j+1}$, etc. on positive j-axis i.e. OY. $U_{I,j-1}, u_{I,j-2}, u_{I,j-3}$, etc on negative j-axis i.e. OY'

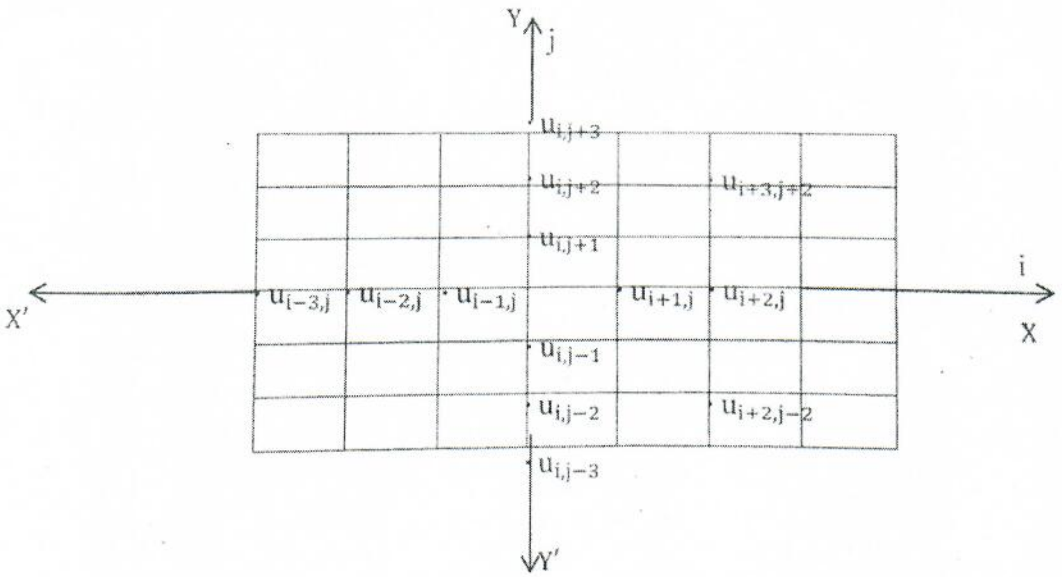

Figure-1 Finite Difference Grid on Rectangular Domain

2.1.1. Jacobi's Iteration Method

Let $u_{i,j}^{(n)}$ be the n^{th} iterative value of $u_{i,j}$. Then Jacobi's iterative procedure is given below.

$$U_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

2.1.2. Gauss-Seidel Method

This method utilizes the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. The formula is given below.

$$U_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

2.1.3. Successive Over-Relaxation or S.O.R Method

Gauss-Seidel formula can be written as

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} + \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)} - 4u_{i,j}^{(n)}\right] = u_{i,j}^{(n)} + \frac{1}{4}R_{i,j}$$

It gives the change $\frac{1}{4}R_{i,j}$ in the value of $u_{i,j}$ for one Gauss- Seidel iteration. In the S.O.R method, larger change than this given to $u_{i,j}^{(n)}$ and the iteration formula is given below.

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} + \frac{1}{4}\omega R_{i,j} = \frac{1}{4}\omega\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right] + (1-\omega)u_{i,j}^{(n)}$$

Here ω is called the accelerating factor and lies between 1 and 2.

3. SOME EXAMPLES OF NUMERICAL RESULTS

3.1. Example

Using Jacobi's method, Gauss-Seidel method and SOR method to solve Laplace's equation

$u_{xx} + u_{yy} = 0, \quad 0 \le x, \ y \ge 1,$ for the boundary values as indicated in Figure.

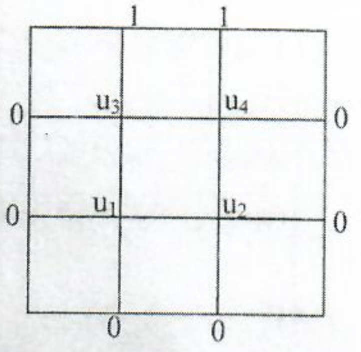

Figure-2

We can solve this example by Jacobi's iteration method:

Let the interior mesh values be u_1, u_2, u_3 and u_4. We need to calculate only u_1, u_2, u_3 and u_4.

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}]$$

where $u_{i,j}^{(n)}$ is the n^{th} iterative value of $u_{i,j}$.

$$u_1^{(n+1)} = \frac{1}{4}[0 + 0 + u_2^{(n)} + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[0 + 0 + u_1^{(n)} + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[1 + 0 + u_1^{(n)} + u_4^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[1 + 0 + u_2^{(n)} + u_3^{(n)}].$$

The iterations have been continued using Ms-Excel and successive iterative are given as follows:

Jacobi's Iterative Method				
	u1	u2	u3	u4
n=1	0.250000	0.250000	0.500000	0.500000
n=2	0.187500	0.187500	0.437500	0.437500
n=3	0.156250	0.156250	0.406250	0.406250
n=4	0.140625	0.140625	0.390625	0.390625
n=5	0.132813	0.132813	0.382813	0.382813
n=6	0.128906	0.128906	0.378906	0.378906
n=7	0.126953	0.126953	0.376953	0.376953
n=8	0.125977	0.125977	0.375977	0.375977
n=9	0.125488	0.125488	0.375488	0.375488
n=10	0.125244	0.125244	0.375244	0.375244
n=11	0.125122	0.125122	0.375122	0.375122
n=12	0.125061	0.125061	0.375061	0.375061
n=13	0.125031	0.125031	0.375031	0.375031
n=14	0.125015	0.125015	0.375015	0.375015
n=15	0.125008	0.125008	0.375008	0.375008
n=16	0.125004	0.125004	0.375004	0.375004
n=17	0.125002	0.125002	0.375002	0.375002
n=18	0.125001	0.125001	0.375001	0.375001
n=19	0.125000	0.125000	0.375000	0.375000
n=20	0.125000	0.125000	0.375000	0.375000
n=21	0.125000	0.125000	0.375000	0.375000

We can solve this example by Gauss-Seidel method:

$$u_1^{(n+1)} = \frac{1}{4}[0+0+u_2^{(n)}+u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[0+0+u_1^{(n+1)}+u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4}[1+0+u_1^{(n+1)}+u_4^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[1+0+u_2^{(n+1)}+u_3^{(n+1)}].$$

The iterations have been continued using Ms-Excel and successive iterative are given as follows:

Gauss's Seidel Method				
	u1	u2	u3	u4
n=1	0.250000	0.312500	0.562500	0.468750
n=2	0.218750	0.171875	0.421875	0.398438
n=3	0.148438	0.136719	0.386719	0.380859
n=4	0.130859	0.127930	0.377930	0.376465
n=5	0.126465	0.125732	0.375732	0.375366
n=6	0.125366	0.125183	0.375183	0.375092
n=7	0.125092	0.125046	0.375046	0.375023
n=8	0.125023	0.125011	0.375011	0.375006
n=9	0.125006	0.125003	0.375003	0.375001
n=10	0.125001	0.125001	0.375001	0.375000
n=11	0.125000	0.125000	0.375000	0.375000
n=12	0.125000	0.125000	0.375000	0.375000
n=13	0.125000	0.125000	0.375000	0.375000
n=14	0.125000	0.125000	0.375000	0.375000
n=15	0.125000	0.125000	0.375000	0.375000
n=16	0.125000	0.125000	0.375000	0.375000
n=17	0.125000	0.125000	0.375000	0.375000

We can solve this example by SOR method with $\omega = 1.1$:

$$u_1^{(n+1)} = \frac{\omega}{4}[0+0+u_2^{(n)}+u_3^{(n)}]+(1-\omega)u_1^{(n)}$$

$$u_2^{(n+1)} = \frac{\omega}{4}[0+0+u_1^{(n+1)}+u_4^{(n)}]+(1-\omega)u_2^{(n)}$$

$$u_3^{(n+1)} = \frac{\omega}{4}[1+0+u_1^{(n+1)}+u_4^{(n)}]+(1-\omega)u_3^{(n)}$$

$$u_4^{(n+1)} = \frac{\omega}{4}[1+0+u_2^{(n+1)}+u_3^{(n+1)}]+(1-\omega)u_4^{(n)}.$$

The iterations have been continued using Ms-Excel and successive iterative are given as follows:

SOR Method

	u1	u2	u3	u4
n=1	0.275	0.350625	0.525625	0.41596875
n=2	0.21346875	0.138032813	0.395532813	0.380133672
n=3	0.125383672	0.125213988	0.374463988	0.374398076
n=4	0.124873076	0.124778168	0.374853168	0.37495881
n=5	0.12491131	0.124986466	0.374978966	0.374994613
n=6	0.124999363	0.124999697	0.375000447	0.375000578
n=7	0.125000103	0.125000218	0.375000143	0.375000041
n=8	0.125000089	0.125000014	0.375000022	0.375000006
n=9	0.125000001	0.125	0.375	0.374999999
n=10	0.125	0.125	0.375	0.375
n=11	0.125	0.125	0.375	0.375
n=12	0.125	0.125	0.375	0.375
n=13	0.125	0.125	0.375	0.375
n=14	0.125	0.125	0.375	0.375
n=15	0.125	0.125	0.375	0.375
n=16	0.125	0.125	0.375	0.375

We conclude that $u_1 = 0.125$, $u_2 = 0.125$, $u_3 = 0.375$, $u_4 = 0.375$.

3.2. Example

We can solve the Poisson's equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$; $x = 3 = y$ with $u(x, y) = 0$ on the boundary and mesh-length is 1.

Draw the figure with sides $x = 0 = y$; $x = 3 = y$ and $h = 1$.

Let u_1, u_2, u_3, u_4 be the values of u at the internal mesh points. i.e., u_1 be the average temperature at $x = 1$ and $y = 2$.

The standard five point formula for the given partial differential equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2 + j^2 + 10) .$$

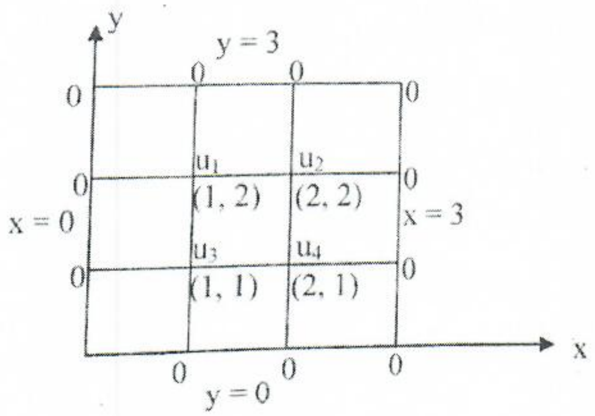

Figure-3

We can solve this example by Jacobi's iteration method:

$$u_1^{(n+1)} = \frac{1}{4}(u_2^{(n)} + u_3^{(n)} + 150)$$

$$u_2^{(n+1)} = \frac{1}{4}(2u_1^{(n)} + 180)$$

$$u_3^{(n+1)} = \frac{1}{4}(2u_1^{(n)} + 120).$$

Jacobi's Iterative Method			
	u1	u2	u3
	37.500000	63.750000	48.750000
n=1	65.625000	63.750000	48.750000
n=2	65.625000	77.812500	62.812500
n=3	72.656250	77.812500	62.812500
n=4	72.656250	81.328125	66.328125
n=5	74.414063	81.328125	66.328125
n=6	74.414063	82.207031	67.207031
n=7	74.853516	82.207031	67.207031
n=8	74.853516	82.426758	67.426758
n=9	74.963379	82.426758	67.426758
n=10	74.963379	82.481689	67.481689
n=11	74.990845	82.481689	67.481689
n=12	74.990845	82.495422	67.495422
n=13	74.997711	82.495422	67.495422
n=14	74.997711	82.498856	67.498856
n=15	74.999428	82.498856	67.498856
n=16	74.999428	82.499714	67.499714
n=17	74.999857	82.499714	67.499714
n=18	74.999857	82.499928	67.499928
n=19	74.999964	82.499928	67.499928
n=20	74.999964	82.499982	67.499982
n=21	74.999991	82.499982	67.499982
n=22	74.999991	82.499996	67.499996
n=23	74.999998	82.499996	67.499996
n=24	74.999998	82.499999	67.499999
n=25	74.999999	82.499999	67.499999
n=26	74.999999	82.500000	67.500000
n=27	75.000000	82.500000	67.500000
n=28	75.000000	82.500000	67.500000
n=29	75.000000	82.500000	67.500000
n=30	75.000000	82.500000	67.500000

We can solve this example by Gauss's Seidel iteration method:

$$u_1^{(n+1)} = \frac{1}{4}(u_2^{(n)} + u_3^{(n)} + 150)$$

$$u_2^{(n+1)} = \frac{1}{4}(2u_1^{(n+1)} + 180)$$

$$u_3^{(n+1)} = \frac{1}{4}(2u_1^{(n+1)} + 120).$$

Gauss's Seidel Method			
	u1	u2	u3
	37.500000	63.750000	48.750000
n=1	65.625000	77.812500	62.812500
n=2	72.656250	81.328125	66.328125
n=3	74.414063	82.207031	67.207031
n=4	74.853516	82.426758	67.426758
n=5	74.963379	82.481689	67.481689
n=6	74.990845	82.495422	67.495422
n=7	74.997711	82.498856	67.498856
n=8	74.999428	82.499714	67.499714
n=9	74.999857	82.499928	67.499928
n=10	74.999964	82.499982	67.499982
n=11	74.999991	82.499996	67.499996
n=12	74.999998	82.499999	67.499999
n=13	74.999999	82.500000	67.500000
n=14	75.000000	82.500000	67.500000
n=15	75.000000	82.500000	67.500000
n=16	75.000000	82.500000	67.500000

We can solve this example by SOR method with $\omega = 1.1$:

$$u_1^{(n+1)} = \frac{\omega}{4}(u_2^{(n)} + u_3^{(n)} + 150) + (1-\omega)u_1^{(n)}$$

$$u_2^{(n+1)} = \frac{\omega}{4}(2u_1^{(n+1)} + 180) + (1-\omega)u_2^{(n)}$$

$$u_3^{(n+1)} = \frac{\omega}{4}(2u_1^{(n+1)} + 120) + (1-\omega)u_3^{(n)}.$$

SOR Method			
	u1	u2	u3
	37.5	70.125	53.625
n=1	71.53125	81.829688	66.979688
n=2	75.019453	82.57773	67.56273
n=3	75.036681	82.512402	67.513902
n=4	75.003565	82.500721	67.500571
n=5	74.999999	82.499927	67.499942
n=6	74.999964	82.499988	67.499986
n=7	74.999996	82.499999	67.499999
n=8	75	82.5	67.5
n=9	75	82.5	67.5
n=10	75	82.5	67.5

We conclude that $u_1 = 75,\ u_2 = 82.5,\ u_3 = 67.5.$

4. CONCLUSION

In this paper, we emphasized on the numerical solutions of Elliptic problems using five points finite difference approximation. The generated linear system is then solved by basic iterative methods namely; Jacobi, Gauss-Seidel and Successive Over Relaxation methods (SOR). We also observed that, Jacobi, Gauss-Seidel and SOR methods are easy to implement. But, impractical for problems with large number of grids and also Successive Over Relaxation requires the optimum value of relaxation parameter for fast convergence, which needs on extra computation. Thus, the SOR could be considered more efficient of three methods for small grids. The solution techniques used above can be greatly extended many other types of equations. In many cases analytical solutions are not enough thus we rely on numerical solutions obtain more information on the inherent problems. But we need to undertake more research on this topic to further our knowledge so that we can effectively utilized our limited resources for the betterment of humanity.

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REFERENCES

[1] Cooper, Jeffery(1998). Introduction to Partial Differential Equations with MATLAB, Birkhauser.

[2] Polking, John, Albert, Bogges, Dave, Arnold(2002), Differential Equations, Prentice Hall.

[3] S.S.Sastry(2011), Advanced Engineering Mathematic, New Delhi.